JENN

Training and Consultancy

The path to enlightened education

SUBJECT: MATHEMATICS

CONTENT MANUAL

ALGEBRAIC FUNCTIONS AND INVERSES

2024
GRADE 12

- LINEAR FUNCTION
- HYPERBOLIC FUNCTION
- QUADRATIC FUNCTION
- EXPONENTIAL FUNCTION
- TRANSFORMATION OF FUNCTIONS
- INVERSES OF FUNCTION
- COMBINATIONS
- AVERAGE GRADIENT BETWEEN TWO POINTS

JENN MAREMATLOU

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FUNCTIONS

Outcomes:

Revise the effect of the parameters a and q and investigate the effect of p on the graphs of the functions defined by: 1.1. $y = f(x) = a(x+p)^2 + q$ 1.2. $y = f(x) = \frac{a}{x+p} + q$ 1.3. $y = f(x) = ab^{x+p} + q$ where $b > 0, b \neq 1$	Investigate numerically the average gradient between two points on a curve and develop an intuitive understanding of the concept of the gradient of a curve at a point.	1. Definition of a *function*. 2. General concept of the *inverse of a function* and how the domain of the function may need to be restricted (in order to obtain a one-to-one function) to ensure that the inverse is a function. 3. Determine and sketch graphs of the inverses of the functions defined by $y = ax + q$; $y = ax^2$ $y = b^x; (b > 0, b \neq 1)$ Focus on the following characteristics: domain and range, intercepts with the axes, turning points, minima, maxima, asymptotes (horizontal and vertical), shape

(SOURCE: CURRICULUM AND ASSESSMENT POLICY STATEMENT(CAPS)SENIOR PHASE GRADES (10 – 12) MATHEMATICS)

The domain is the set of allowed x-values which determine the range of y-values.

Important terminilogy

Domain:	the set of possible x-values	**For all functions**
Range:	the set of possible y-values	
Axis of symmetry:	an imaginary line that divides a graph into two mirror images of each other.	**See the hyperbola and parabola**
Maximum:	the highest possible y-value of a function.	**See the parabola**
Minimum:	the lowest possible y-value of a function.	
Asymptote:	an imaginary line that a graph approaches but never touches.	**See the hyperbola and exponential function**
Turning point:	The point at which a graph reaches its maximum or minimum value and changes direction.	**See the parabola**

The concepting of increasing and decresing in functions: all functions

- The function is INCREASING when the value of y increases as x is increasing from left to right
  - **THE GRAPH GOES UP**

- The function is DECREASING when the value of y decreases as x is increasing from left to right
  - **THE GRAPH GOES DOWN**

LINEAR FUNCTION (STRAIGHT LINE)

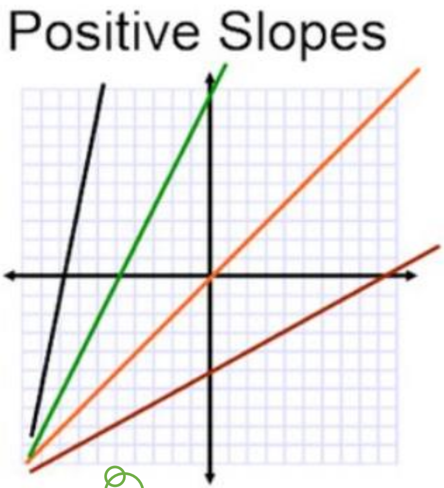
The graph of $y = mx + c$

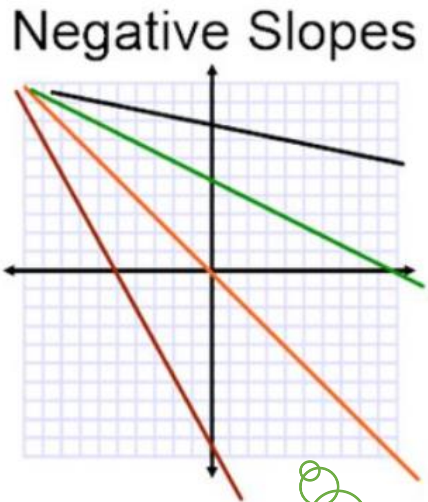
Standard form of linear function
WHERE
$m = gradient$
$c = y - intercept$

When $m > 0$ ***(gradient is positive)*** and
$m < 0$ ***(gradient is negative)***

Domain: $x \in R$
Range: $y \in R$

Shape





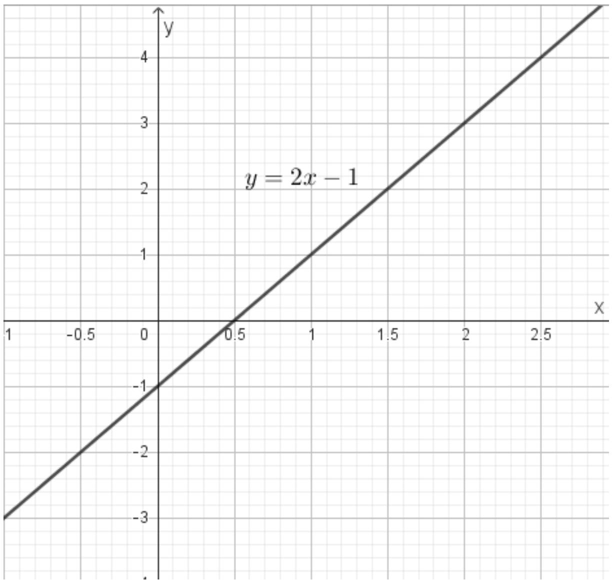
When $m > 0$
1. **The gradient is positive**
2. **The function is increasing**

When $m < 0$
3. **The gradient is negative**
4. **The function is decreasing**

Example 1

Sketch the graph of $y = 2x - 1$ and determine the domain and range of the function, and state if the function is increasing or decreasing.

Solution



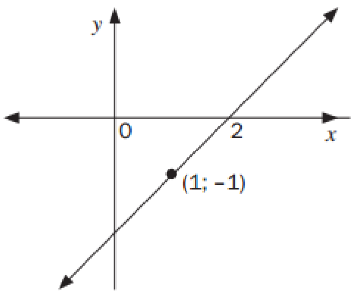
$\boldsymbol{x - intercept: ley\ y = o}$
$0 = 2x - 1$
$-2x = -1$
$x = \frac{1}{2}$
$\therefore \left(\frac{1}{2}; 0\right)$
$y - intercept: let\ x = 0$
$y = 2(0) - 1$
$y = -1$
$\therefore (0; -1)$

Domain: $x \in R$
Range: $y \in R$

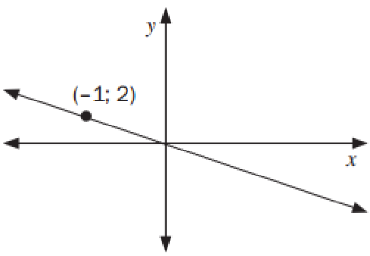
The function is increasing

Example 2
Determine the equations of the following graphs:

1.



2.



Solutions

1.

$a = \frac{y_2 - y_1}{x_2 - x_1}$
$= \frac{-1 - 0}{1 - 2}$
$a = 1$
$\therefore y = 1x + c$
$0 = 1(2) + c$
$c = -2$
$y = x - 2$

2.

$a = \frac{y_2 - y_1}{x_2 - x_1}$
$= \frac{2 - 0}{-1 - 0}$
$a = -2$
$\therefore y = -2x + c$
$0 = -2(0) + c$
$c = 0$
$y = -2x$

HYPERBOLIC FUNCTIONS(HYPERBOLA)

The graph of $y = \frac{a}{x+p} + q$ Standard form of hyperbola

take note that $\boldsymbol{y = \frac{2}{x-2} + 1}$

$$\boldsymbol{= \frac{2}{x+(-2)} + 1}$$

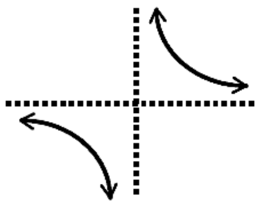
The equations of asymptotes are $x = -p$ ***(vertical asymptote)*** and
$y = q$ ***(horizontal asymptote)***

Domain: $x \in R, x \neq -p$

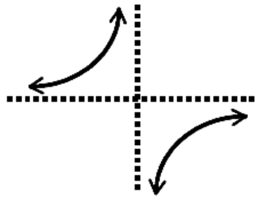
Range: $y \in R, y \neq q$

Shape

If $a > 0$ then the graph decreases for all $x < 0$ or $x > 0$.



If $a < 0$ then the graph increases for all $x < 0$ or $x > 0$.



The equations of the axis of symmetry

The hyperbola has two equations of symmetry

$\boldsymbol{m = 1}$	$\boldsymbol{m = -1}$
$\boldsymbol{y = x + c}$	$\boldsymbol{y = -x + c}$

N.B the equations of the axis of symmetry of the hyperbola passes through the point of intersection of asymptotes $(-p; q)$

In general, for the hyperbola, the equations of the axis of symmetry are given by the following formulae:

$\boldsymbol{m = 1}$	$\boldsymbol{m = -1}$
$\boldsymbol{y = (x + p) + q}$ $\boldsymbol{\therefore y = x + p + q}$	$\boldsymbol{y = -(x + p) + q}$ $\boldsymbol{\therefore y = -x - p + q}$

N.B Ensure that the hyperbola is in standard form before applying the formula

Example 1

Sketch the graph of $y = \frac{10}{x+2} - 3$, write down the domain and range of the function, and write down the equations of asymptotes. state whether the function is increasing or decreasing. Lastly, determine the equation of the asymptote with a positive gradient.

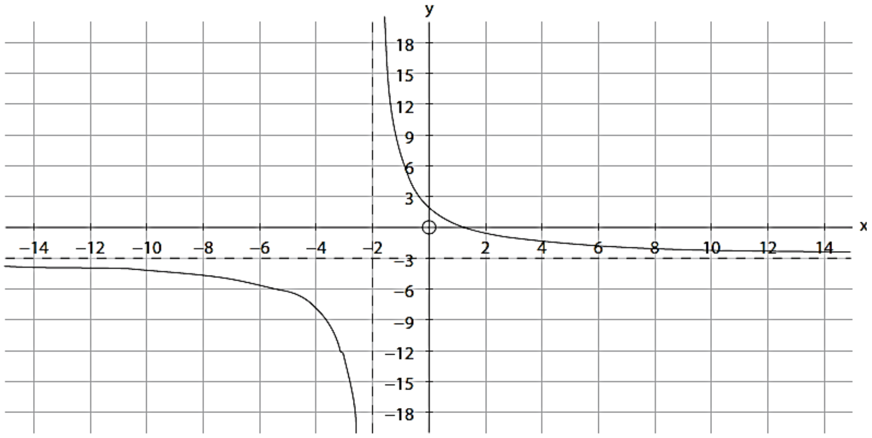
Solutions

The asymptotes are $x = -2$ and $y - 3$. (These are read from the equation).

x-intercept: Let $y = 0$:

$0 = \frac{10}{x+2} - 3 \therefore 10 = 3(x+2) \quad \therefore 3x = 4 \quad \therefore x = \frac{4}{3}$

y-intercept: Let $x = 0$: $y = \frac{10}{0+2} - 3 = 5 - 3 = 2$



graph not drawn to scale

Domain: $x \in R, x \neq -2$

Range: $y \in R, y \neq -3$

The function is decreasing

<u>Equation of the axis of symmetry with positive gradient:</u>

$y = x + c$

$-3 = -2 + c$

$c = -1$

$\therefore y = x - 1$

Example 2

Sketch the graph of $y = -1 - \frac{8}{x-4}$, and write down the domain and range of the function. And determine the equation of the axis of symmetry with a negative gradient.

Solution

This equation can also be written as $y = \frac{-8}{x-4} - 1$.

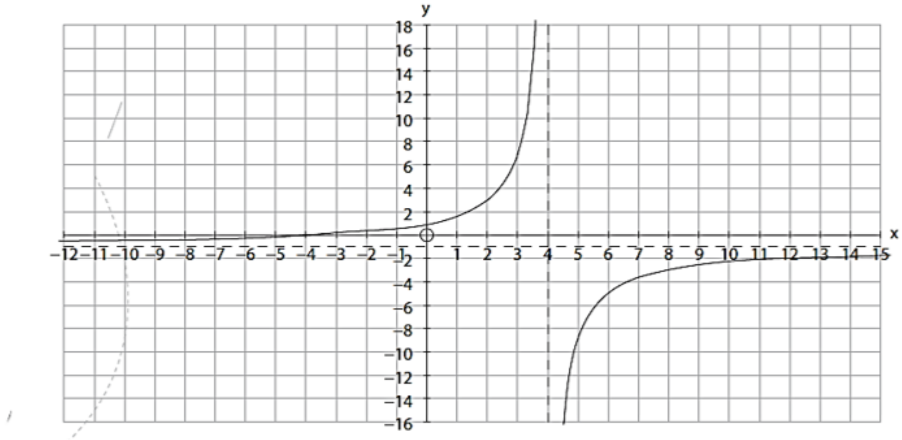
Standard form of hyperbola

Asymptotes: $x = 4$ and $y = -1$

y-intercept: Let $x = 0$: $y = -1 - \frac{8}{0-4} = -1 + 2 = 1$

x-intercept: Let $y = 0$: $0 = -1 - \frac{8}{x-4}$ $\quad \therefore (x-4) = -8 \quad \therefore x = -4$

The hyperbola will look as follows:



graph not drawn to scale

Domain: $x \in R, x \neq 4$
Range: $y \in R, y \neq -1$

The function is increasing

<u>Equation of the axis of symmetry with positive gradient:</u>
$y = -x + c$
$-1 = -(4) + c$
$c = 3$
$\therefore y = -x + 3$

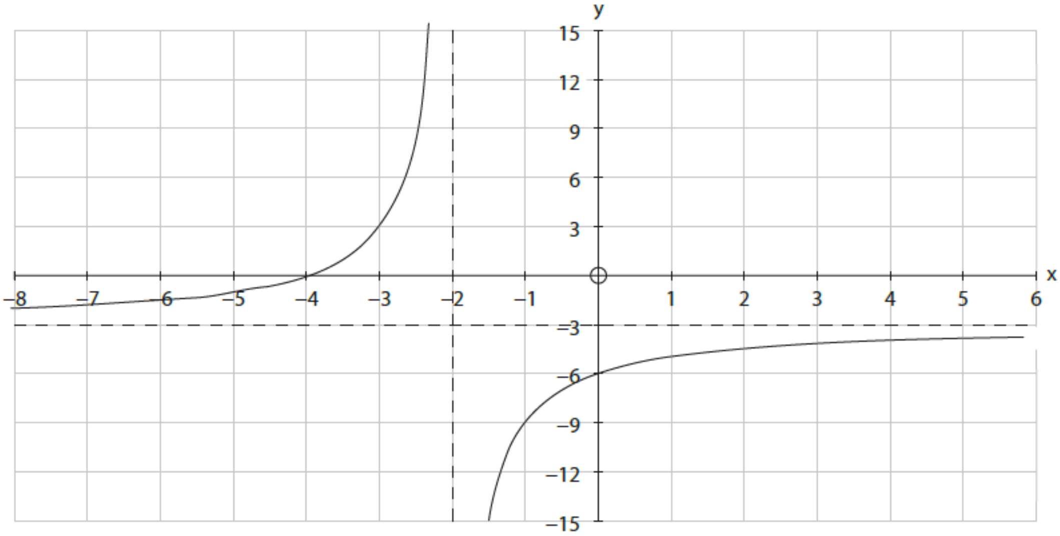
Example 3

Determining the equation of a hyperbola when the graph is given

Step 1: The position of the asymptotes give us the value(s) of p and q in $y = \frac{a}{x+p} + q$.

Step 2: To find the value of a, we substitute any point on the graph into the equation.

EXAMPLE: The graph of $y = \frac{a}{x+p} + q$ is sketched below. Determine the value(s) of a, p and q.



From the graph we see that, $p = 2$ and $q = -3$.

The equation becomes: $y = \frac{a}{x+2} - 3$.

Substituting (0; –6) or any other point on the graph into the equation gives:

$-6 = \frac{a}{0+2} - 3 \quad \therefore -12 = a - 6 \quad \therefore a = -6$

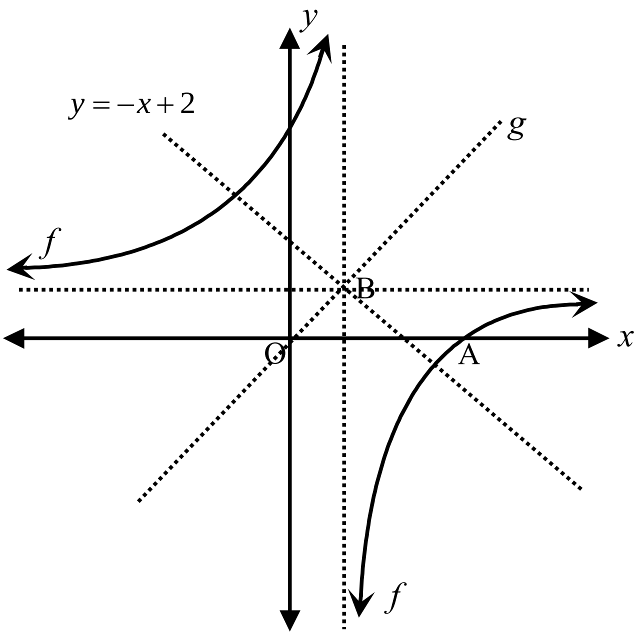
$$\therefore \boldsymbol{y = -\frac{6}{x+2} - 3}$$

$\boldsymbol{N.B\ the\ function\ is\ increasing}$

Activities

The diagram below shows the hyperbola defined by $f(x)=\dfrac{-4}{x+p}+1$

The lines g and $y=-x+2$ are axes of symmetry of f and intersect at B, the point of intersection of the asymptotes.



1.1 Determine the equation of f and hence the value of p.

1.2 Determine the equation of g, the other axis of symmetry of f.

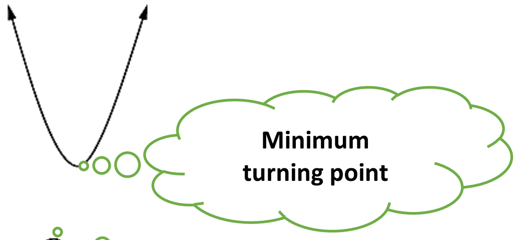
1.3 Write down the domain of f.

1.4 Suppose that the graph of f is shifted left so that A coincides with the origin. Determine the equation of the vertical asymptote of the newly formed graph.

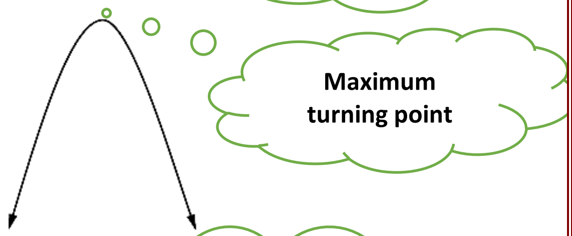
QUADRATIC FUNCTION(PARABOLA)

The graph of $y = a(x+p)^2 + q$

If a is positive, i.e. a > 0, then the shape of the graph is ☺.



If a is negative, i.e. a < 0, then the shape of the graph is ☹.



The graph has the axis of symmetry at $x = -p$

The graph has the turning point by $(-p\ ;\ q)$

Domain: $x \in R$

Range: $y \geq q)$ 😊 **(WHEN a > 0)**
or
$y \leq q$ 😟 **(WHEN a < 0)**

N.B
1. q is the minimum of the parabola when $a > 0$

2. q is the maximum of the parabola when $a < 0$

3. We obtain the maximum or the minimum at $x = -p$

N.B The parabola changes from increasing to decreasing or decreasing to increasing at the turning point.

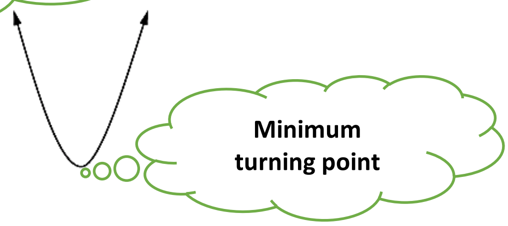
<u>when a > 0</u> $\mathbf{1.}\ \boldsymbol{The\ graph\ increases\ for: x > -p}$ $\mathbf{2.}\ \boldsymbol{The\ graph\ decrease\ for: x < -p}$
<u>when a < 0</u> $\mathbf{1.}\ \boldsymbol{The\ graph\ increases\ for: x < -p}$ $\mathbf{2.}\ \boldsymbol{The\ graph\ decrease\ for: x > -p}$

The quadratic function can also be represented in the form:

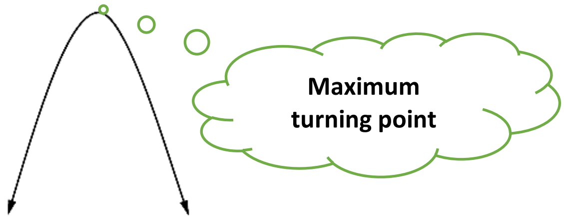
$y = f(x) = ax^2 + bx + c$

Standard form of parabola

If a is positive, i.e. a > 0, then the shape of the graph is ☺.



If a is negative, i.e. a < 0, then the shape of the graph is ☹.



The graph has the axis of symmetry at $x = -\frac{b}{2a}$

The graph has the turning point by $(-\frac{b}{2a}\ ;\ f(-\frac{b}{2a}))$

Domain: $x \in R$

Range: $y \geq f(-\frac{b}{2a})$ 😊 **(WHEN a > 0)**

or

$y \leq f(-\frac{b}{2a})$ 😟 **(WHEN a < 0)**

N.B

1. $\boldsymbol{f\left(-\frac{b}{2a}\right)}$ **is the minimum of the parabola when** $\boldsymbol{a > 0}$

2. $\boldsymbol{f\left(-\frac{b}{2a}\right)}$ **is the maximum of the parabola when** $\boldsymbol{a < 0}$

3. We obtain the maximum or the minimum at $\boldsymbol{x = -\frac{b}{2a}}$

Example 1

Sketch the graph of $f(x) = x^2 - 5x - 6$, clearly showing intercepts with the axes and the turning point. Also give the interval where the function is increasing and where the function is decreasing. Lastly, determine the domain and range of of $f(x) = x^2 - 5x - 6$.

Solutions

Sketch the graph of $f(x) = x^2 - 5x - 6$

1. ***y*-intercept**

   $f(0) = -6$

   Therefore the co-ordinates of the *y*-intercept are (0; –6)

2. ***x*-intercept**

   $x2 - 5x - 6 = 0$

   $(x - 6)(x + 1) = 0$

   $x = 6$ or $x = -1$

   (6; 0) and (–1; 0)

3. **Axis of symmetry**

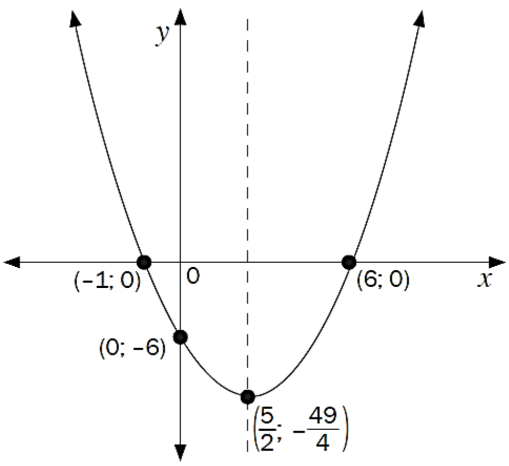
   $$x = \frac{-b}{2a} = \frac{-(-5)}{2(1)} = \frac{5}{2}$$

4. **Turning point**

   $$f\left(\frac{5}{2}\right) = \left(\frac{5}{2}\right)^2 - 5\left(\frac{5}{2}\right) - 6 = -12\frac{1}{4}$$

   $$\therefore TP\left(\frac{5}{2}; -12\frac{1}{4}\right)$$

5. **Sketch graph**



6. The interval where the graph is increasing

$x > \frac{5}{2}$

The interval where the graph is decreasing

$x < \frac{5}{2}$

7. Domain: $x \in R$

Range: $y \geq -\frac{49}{4}$

Determining the equation of the quadratic function

Given the x-intercepts and one point	Given the turning point and one point
• Use the formula: $y = a(x - x_1)(x - x_2)$. • Substitute the values of the x-intercepts. • Substitute the given point which is not the x-intercept. • Solve for a. • Write the equation in the form $f(x) = ax^2 + bx + c$.	• Use the formula: $y = a(x + p)^2 + q$. • Substitute the co-ordinates of the turning point $(p; q)$. • Substitute the given point. • Solve for a. • Write the equation in the form $y = a(x + p)^2 + q$ or $f(x) = ax^2 + bx + c$ depending on the instruction in the question.

Given the co-ordinates of three points on the parabola
• Use the formula: $y = ax^2 + bx + c$. • One of the given point is the y-intercept, therefore c is given, so substitute its value. • Substitute the co-ordinates of the other two points into $y = ax^2 + bx + c$. • Solve the two equations simultaneously for a and b.

SOURCE:Mind The Gap Mathematics Study Guide

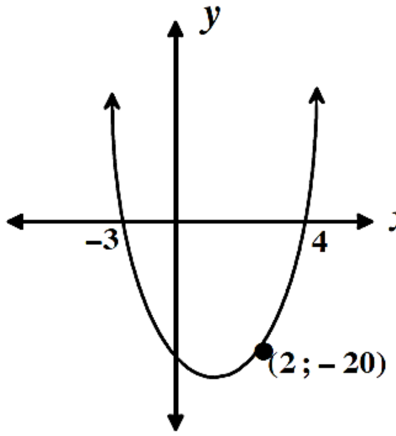
Example 2

Determine the equation of the parabola in the form $f(x) = ax^2 + bx + c$.

$y = a(x - x_1)(x - x_2)$

$\therefore y = a(x - (-3))(x - 4)$

$\therefore y = a(x + 3)(x - 4)$



Substitute $(2\,; -20)$ to find the value of a.

$-20 = a(2 + 3)(2 - 4)$

$\therefore -20 = a(5)(-2)$

$\therefore -20 = -10a$

$\therefore a = 2$

$\therefore y = 2(x + 3)(x - 4)$

$\therefore y = 2(x^2 - x - 12)$

$\therefore f(x) = 2x^2 - 2x - 24$

Example 3

Determine the equation of g in the form $y = ax^2 + bx + c$

The axis of symmetry is given by $x = -1$.

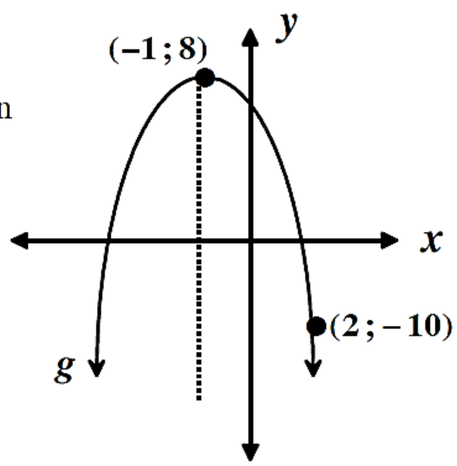
$\therefore x + 1 = 0$

From the work on parabolas, it is clear that the expression $x + 1$ is in the brackets of the equation for a parabola.

Also, the value of q is 8 (the y-value of the turning point).

$\therefore y = a(x + 1)^2 + 8$

Now substitute the point $(2\,; -10)$ which lies on the graph of the parabola.



$-10 = a(2 + 1)^2 + 8$

$\therefore -18 = a(3)^2$

$\therefore -18 = 9a$

$\therefore a = -2$

$\therefore y = -2(x + 1)^2 + 8$

$\therefore y = -2(x^2 + 2x + 1) + 8$

$\therefore y = -2x^2 - 4x - 2 + 8$

$\therefore y = -2x^2 - 4x + 6$

Activities

1.1 Consider the function $f(x) = -(x-2)^2 + 1$

1.1.1 Draw a neat sketch graph indicating the coordinates of the intercepts with the axes, the coordinates of the turning point and the equation of the axis of symmetry.

1.1.2 Write down the range.

1.2 Consider the following functions: $f(x) = x^2 - 2x - 3$ and $g(x) = 2x^2$

1.2.1 Determine the coordinates of the turning point of f.

1.2.2 Write down the x-intercepts of f.

1.2.3 Draw a neat sketch graph of f.

1.2.4 State the minimum value of f.

1.2.5 Write down the equation of the axis of symmetry of g.

1.2.6 If the graph of g is shifted 3 units left and 1 unit upwards, state the equation of the newly formed graph.

EXPONENTIAL FUNCTION

Standard form of exponential function

The graph of $\boldsymbol{y = a.b^{x+p} + q}$ **where** $\boldsymbol{b > 0}$ **and** $\boldsymbol{b \neq 1}$

The equation of an asymptote is $y = q$ ***(horizontal asymptote)***

Domain: $x \in R$

Range: $y > q$ *[if $a > 0$]* or $y < q$ *[if $a < 0$]*

<table>
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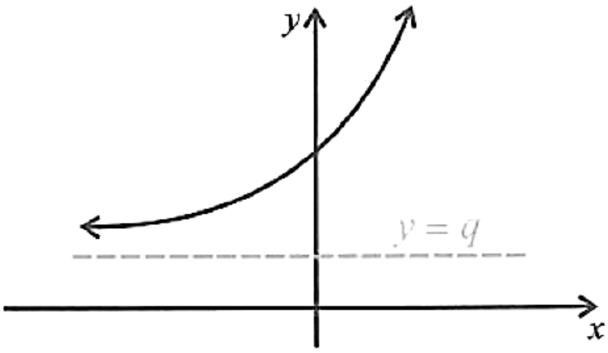
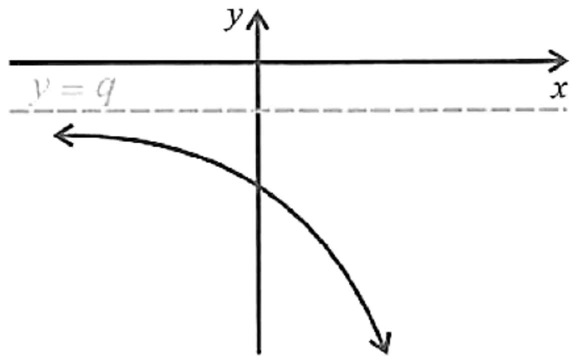
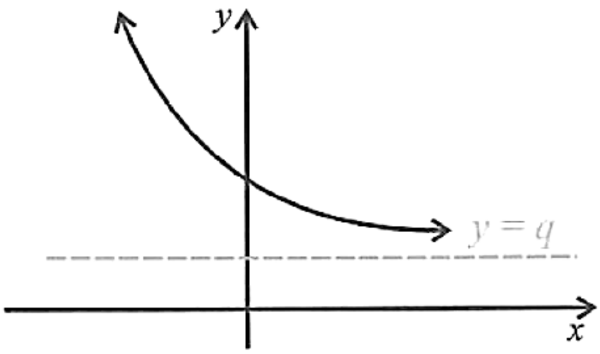
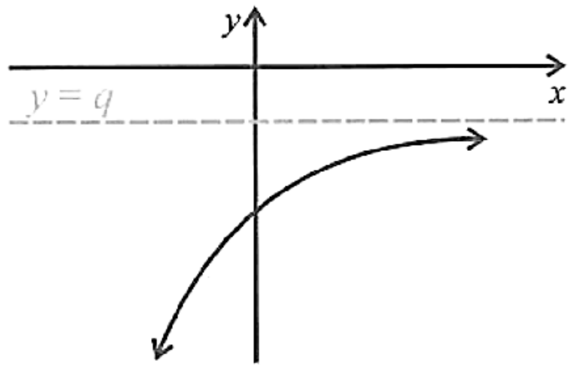
<tr><td>The graph lies above the horizontal asymptote and is an increasing function.
</td><td>The graph lies below the horizontal asymptote and is a decreasing function.
</td></tr>
<tr><th>$a > 0$ and $0 < b < 1$</th><th>$a < 0$ and $0 < b < 1$</th></tr>
<tr><td>The graph lies above the horizontal asymptote and is a decreasing function.
</td><td>The graph lies below the horizontal asymptote and is an increasing function.
</td></tr>
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(Source: MATHS MADE EASY – A comprehensive guide to Grade 12 Mathematics)

Example 1

Sketch the graph of $y = 2\left(\frac{1}{2}\right)^{x+1} - 4$

Horizontal asymptote: $y = -4$

x-intercept: let $y = 0$:

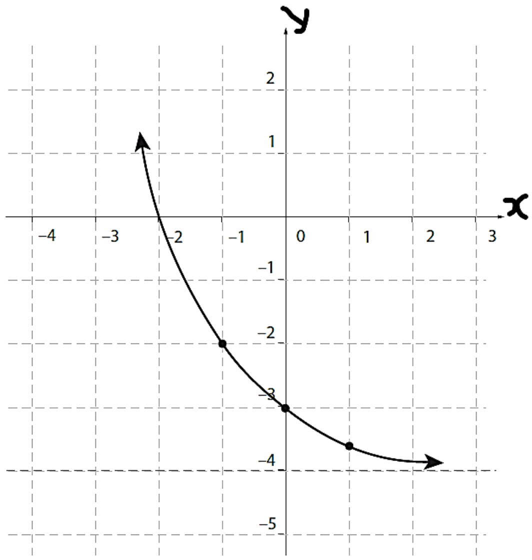
$0 = 2\left(\frac{1}{2}\right)^{x+1} - 4 \therefore \left(\frac{1}{2}\right)^{x+1} = 2$

$\left(\frac{1}{2}\right)^{x+1} = \left(\frac{1}{2}\right)^{-1}$

$\therefore x + 1 = -1 \quad \therefore x = -2$

y-intercept: let x = 0

y = - 3



N.B the function is decreasing

Example 2

Let us look at a second example, $y = 3{\cdot}(3)^{x-2} + 1$

Horizontal asymptote: $y = 1$

x-intercept: $0 = 3{\cdot}(3)^{x-2} + 1$

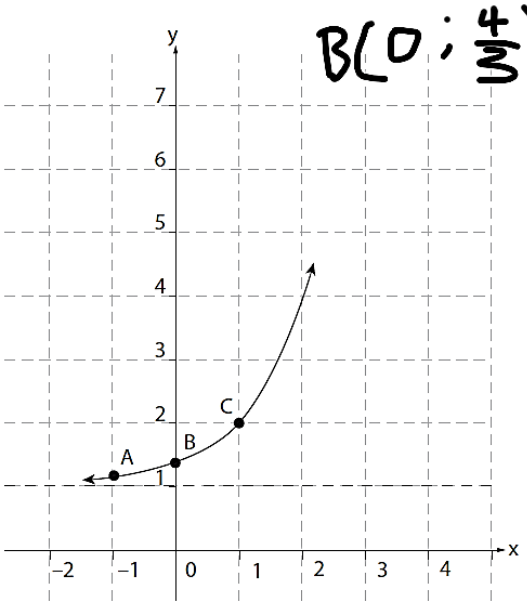
$\therefore 3(3)^{x-2} = -1$

$\therefore$ No x-intercept

y-intercept: let x = 0

y = $\frac{4}{3}$

B$(0; \frac{4}{3})$

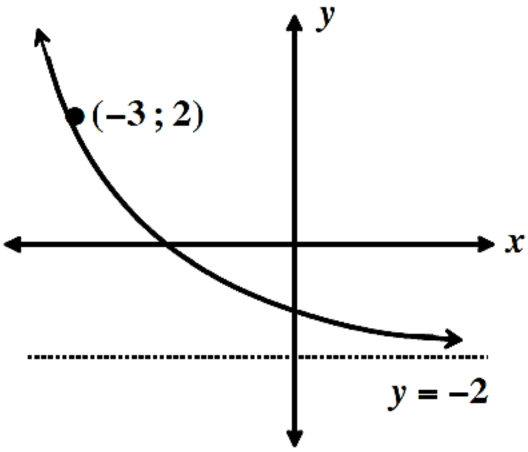


N.B the function is inccreasing

Determining the equation of exponential function

Example 3

Determine the equation of $g(x) = b^{x+1} + q$



Horizontal asymptote: $y = -2$

Therefore the value of q is -2

$\therefore y = b^{x+1} - 2$

Now substitute the point $(-3\,;2)$ into the equation to get b.

$2 = b^{-3+1} - 2$

$\therefore 4 = b^{-2}$

$\therefore 4 = \dfrac{1}{b^2}$

$\therefore 4b^2 = 1$

$\therefore 4b^2 - 1 = 0$

$\therefore (2b+1)(2b-1) = 0$

$\therefore b = -\dfrac{1}{2}$ or $b = \dfrac{1}{2}$

But $b \neq -\dfrac{1}{2}$

$\therefore b = \dfrac{1}{2}$

Therefore the equation is: $g(x) = \left(\dfrac{1}{2}\right)^{x+1} - 2$

Note:
Since $b > 0$, you could have solved the equation $4b^2 = 1$ as follows:

$4b^2 = 1$

$\therefore b^2 = \dfrac{1}{4}$

$\therefore b = \dfrac{1}{2}$

Activities

Consider the following functions:

$$f(x) = 2^{x+1} - 1$$

1.1 Determine the coordinates of the y-intercept of f.

1.2 Determine the coordinates of the x-intercept of f.

2.1.3 Sketch the graph of f.

TRANSFORMATION OF FUNCTIONS

REFLECTIONS AND TRANSLATIONS

Given $f(x) = \dfrac{2}{x+1} - 3$

a. The graph of $g(x)$ is obtained by shifting the graph of $f(x)$ 2 units up and 3 units to left. Determine the equation of $g(x)$.

Solution

$$g(x) = f(x+3) + 2$$
$$= \frac{2}{x+3+1} - 3 + 2$$
$$= \frac{2}{x+4} - 1$$

b. The graph of $h(x)$ is obtained by reflecting the graph of $f(x)$ in the $x-axis$. Determine the equation of $h(x)$.

Solution

$$h(x) = -f(x)$$
$$= -\left(\frac{2}{x+1} - 3\right)$$
$$= -\frac{2}{x+1} + 3$$

c. The graph of $m(x)$ is obtained by reflecting the graph of $f(x)$ in the $y-axis$. Determine the equation of $m(x)$.

Solution

$$m(x) = f(-x)$$
$$= \frac{2}{-x+1} - 3$$
$$= \frac{2}{-(x-1)} - 3$$
$$= -\frac{2}{x-1} - 3$$

Given $f(x) = 2.3^{x-2} + 4$

a. The graph of $g(x)$ is obtained by shifting the graph of $f(x)$ 2 units up and 3 units to left. Determine the equation of $g(x)$.

Solution

$$g(x) = f(x+3) + 2$$
$$= 2.3^{x+3-2} + 4 + 2$$
$$= 2.3^{x+1} + 6$$

b. The graph of $h(x)$ is obtained by reflecting the graph of $f(x)$ in the $x-axis$. Determine the equation of $h(x)$.

Solution

$$h(x) = -f(x)$$
$$= -(2.3^{x-2} + 4)$$
$$= -2.3^{x-2} - 4$$

c. The graph of $m(x)$ is obtained by reflecting the graph of $f(x)$ in the $y-axis$. Determine the equation of $m(x)$.

Solution

$$m(x) = f(-x)$$
$$= 2.3^{-x-2} + 4$$
$$= 2.3^{-(x+2)} + 4$$
$$= 2.\left(\frac{1}{3}\right)^{x+2} + 4$$

Given $f(x) = x^2 + 5x + 6$

a. The graph of $g(x)$ is obtained by shifting the graph of $f(x)$ 2 units down and 3 units to right. Determine the equation of $g(x)$.

Solution

$$g(x) = f(x-3) - 2$$
$$= (x-3)^2 + 5(x-3) + 6 - 2$$
$$= x^2 - 6x + 9 + 5x - 15 + 4$$
$$= x^2 - x - 2$$

b. The graph of $h(x)$ is obtained by reflecting the graph of $f(x)$ in the $x-axis$. Determine the equation of $h(x)$.

Solution

$$h(x) = -f(x)$$
$$= -(x^2 + 5x + 6)$$
$$= -x^2 - 5x - 6$$

c. The graph of $m(x)$ is obtained by reflecting the graph of $f(x)$ in the $y-axis$. Determine the equation of $m(x)$.

Solution

$$m(x) = f(-x)$$
$$= (-x)^2 + 5(-x) + 6$$
$$= x^2 - 5x + 6$$

INVERSE FUNCTIONS

The concept of a function

A function f, is defined as a relationship between values, where each input value maps to one output value.
In other words, for an equation to be called a function, there can only be one y - value for a particular x-value.

There are two types of functions:

1. **One-to-One Functions**
2. **Many-to-One Functions**

ONE-TO-ONE FUNCTIONS

A one-to-one **function is a function where there is a single y-value for a particular x-value.**

MANY-TO-ONE FUNCTIONS

A function cannot have more than one y value to each x value. However, a function can have more than one x value for a particular y value. **These are known as** many to-one **functions.**

VERTICAL LINE TEST

To test if a graph is a function, use the vertical line **test**. If a vertical line (a line parallel to the y-axis) touches the graph more than once at any point, the graph is not a function. You don't have to draw a line, just hold a ruler parallel to the y-axis and move it along the graph. If the ruler touches the graph more than once for a single x value, anywhere on the graph, then the graph is not a function. In the case of the graph not being a function, it is said to be a relation.

HORIZONTAL LINE TEST

If a graph passes the vertical line test, it is a function. **The** horizontal line **test can be used to determine what type of function the graph represents.**

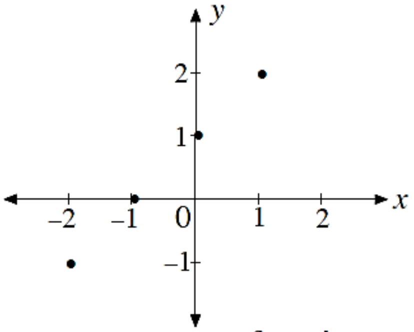
If a horizontal line (a line parallel to the x-axis) is drawn and moved along the graph and it **touches the graph more than once** at any point, it is a **many-to-one function** (many x-values to a single y-value). Otherwise, it is a one-to-one function.

(Source: MATHS MADE EASY – A comprehensive guide to Grade 12 Mathematics)

Examples

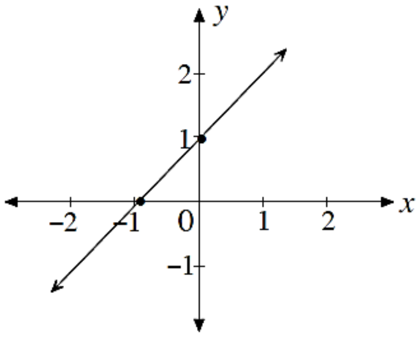
Determine whether the following relations are functions or not. If the graph is a function, determine whether the function is one-to-one or many-to-one.

(a)



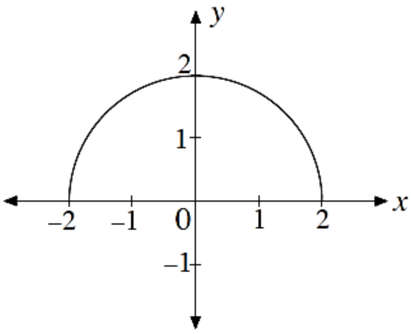
one-to-one function

(b)



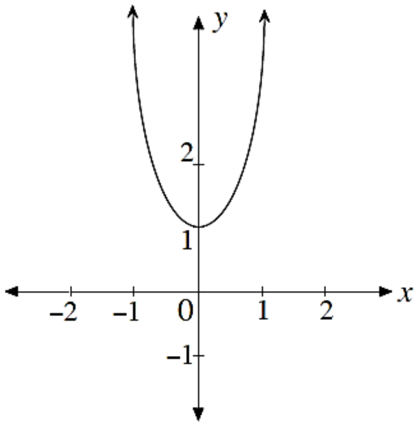
one-to-one function

(c)



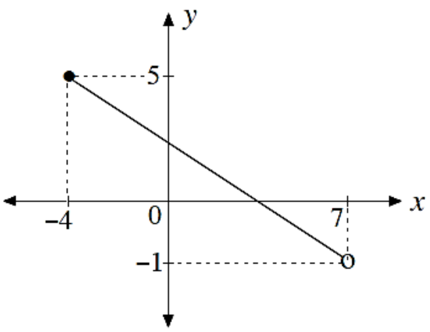
many-to-one function

(d)



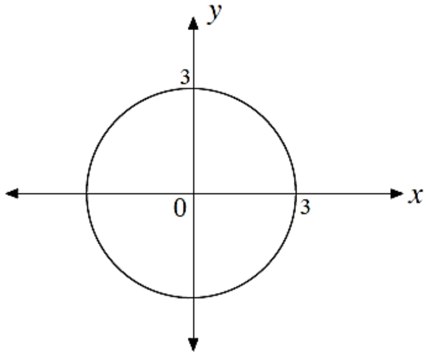
many-to-one function

(e)



one-to-one function

(f)



not a function

(Source: Mind Action series Mathematics12 textbook and Workbook **)**

Inverse Function

- The inverse of a function takes the y-values (range) of the function to the corresponding x-values (domain) and vice versa. Therefore the x and y values are interchanged.
- The function is reflected along the line $y = x$ to form the inverse.
- The notation for the inverse of a function is f^{-1}.

- N.B The domain of the inverse is the range of the function and the range of the inverse is the domain of the function.
- When the function is increasing, its inverse also increases. When the function decreases, its inverse will also decrease.

Inverse function : Linear $(y = ax + q)$

Example 1

Given $f(x) = 2x + 6$.

1. Determine $f^{-1}(x)$

2. Sketch the graphs of $f(x)$, $f^{-1}(x)$ and $y = x$ on the same set of axis

Solutions

1. In order to find the inverse of a function, there are two steps:

STEP 1: Swap the x and y

$$y = 2x + 6$$

becomes $x = 2y + 6$

We then rewrite the equation to make y the subject of the formula.
Therefore,

STEP 2: make y the subject of the formula

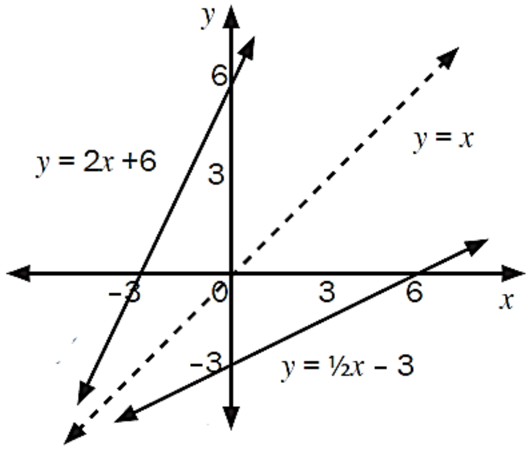
$x = 2y + 6$

$x - 6 = 2y$

So $y = \frac{1}{2}x - 3$

We can say that the inverse function $f^{-1}(x) = \frac{1}{2}x - 3$

2.



Inverse function : Quadratic ($y = ax^2$)

Example 2

a) Sketch $f(x) = 2x^2$

b) Determine the inverse of $f(x)$

c) Sketch $f^{-1}(x)$ and $y = x$ on the same axes as $f(x)$

a) & c)

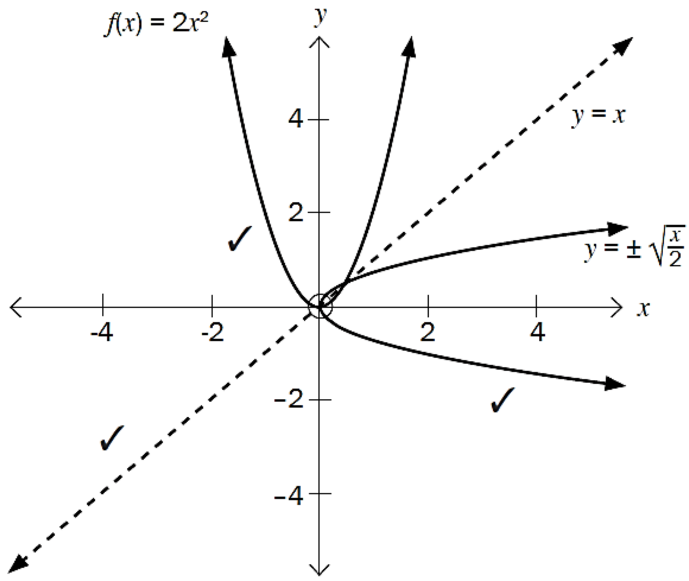
Solution

1. b) $y = 2x^2$

$x = 2y^2$ ✓

$y = \pm\sqrt{\frac{x}{2}}$ ✓

- This is not a function.
- Check it with a vertical line test. There are two y-values for one x-value.
- Not all inverses of functions are also functions. Some inverses of functions are relations.
- If an inverse is not a function, then we can restrict the **domain** of the **function** in order for the inverse to be a function.



- To make the inverse a function, we need to choose a set of x-values in the function and work only with those. We call this **'restricting the domain'**.
- A one to one function has an inverse that is a function

  Example: $y = 3x + 4$ is a one to one function. For every x value there is one and only one y value

- A many to one function has an inverse that is not a function. However, we can restrict the domain of the function to make its inverse a function.

  Example: $y = 2x^2$ is a many to one function. For two or many x values there is one y value. (if $x = 2$, then $y = 8$.
  If $x = -2$, then $y = 8$). Therefore, its inversey $= \pm\sqrt{\frac{x}{2}}$, is not a function.
- To check for a function, draw a vertical line. If any vertical line cuts the graph in only one place, the graph is a function.
  If any vertical line cuts the graph in more than one place, then the graph is not a function.
- To check for a one-to-one function, draw a horizontal line. If any horizontal line cuts the graph in only one place, the graph is a one-to-one function. If any horizontal line cuts the graph in more than one place, then the graph is a many-to-one function.

Example 3

Given: $g(x) = -x^2$ where $x \leq 0$

(a) Write down the inverse of g, g^{-1} in the form $h(x) =$ ....

(b) Sketch the graphs of g, h and $y = x$ on the same set of axis.

Solutions

(a) $y = -x^2$

$x = -y^2$

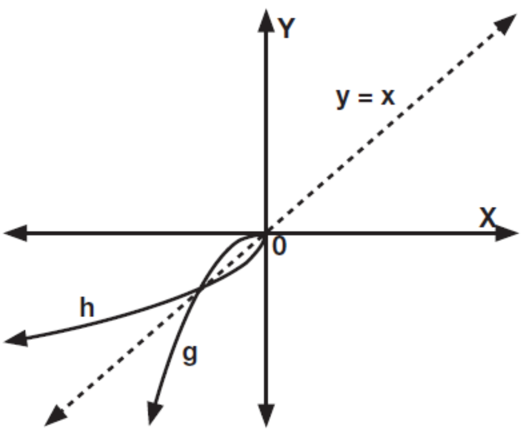
$-x = y^2$

$\pm\sqrt{-x} = y$

$-\sqrt{-x} = y$ where $x \leq 0$

$\therefore h(x) = -\sqrt{-x}$

(b)



Inverse function : Exponential $y = b^x; (b > 0, b \neq 1)$

Example 4

Given: $f(x) = 2^x$

a) Determine f^{-1} in the form $y = \ldots\ldots$

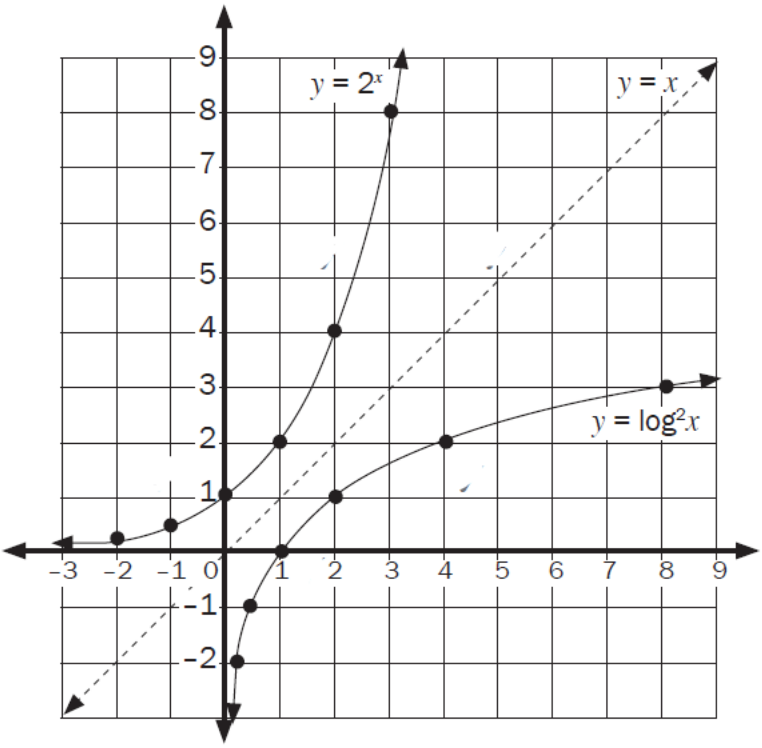
b) Sketch the graphs of $f(x)$, $f^{-1}(x)$ and $y = x$ on the same set of axes.

c) Write the domain and range of $f(x)$ and $f^{-1}(x)$

Solution

a) The inverse of the exponential function $y = 2^x$ is $x = 2^y$ which can be written as $y = \log_2 x$.

b)



c) The domain and Range of $f(x)$

Domain: $x \in R$
Range: $y > 0$

The domain and Range of $f^{-1}(x)$

Domain: $x > 0$
Range: $y \in 0$

Activities

QUESTION 1

Given: $f(x) = 2x^2$ for the domain $x \geq 0$.

2.4 Sketch the graph of f.

1.2 Determine the equation of the inverse function of f.

1.3 Sketch the graph of $y = f^{-1}(x)$ on the same set of axes as f.

QUESTION 2

Given: $f(x) = 3^x$

2.1 Write down the equation of f^{-1} in the form $y = ...$

2.2 Sketch the graph of f and f^{-1} on the same set of axes.

2.3 Solve the following using the graphs of f and f^{-1}:

2.3.1 $f(x).f^{-1}(x) \leq 0$

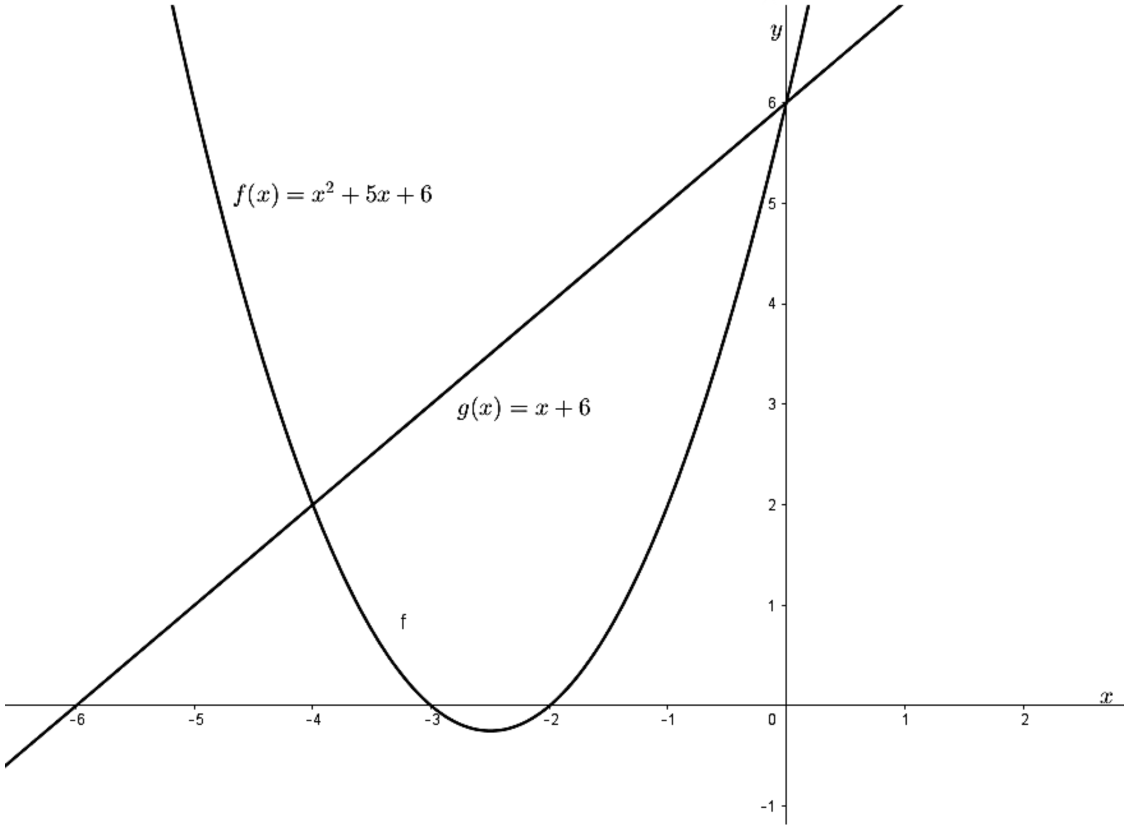
2.3.2 $x.f^{-1}(x) > 0$

2.4 Sketch the graph of $y = g(x) = -f(x-1)+1$ on a set of axes.
Show intercepts with the axes and the asymptote of the graph.

COMBINATIONS

Take note of the following when working with combination of functions:

Consider the graphs of f and g below

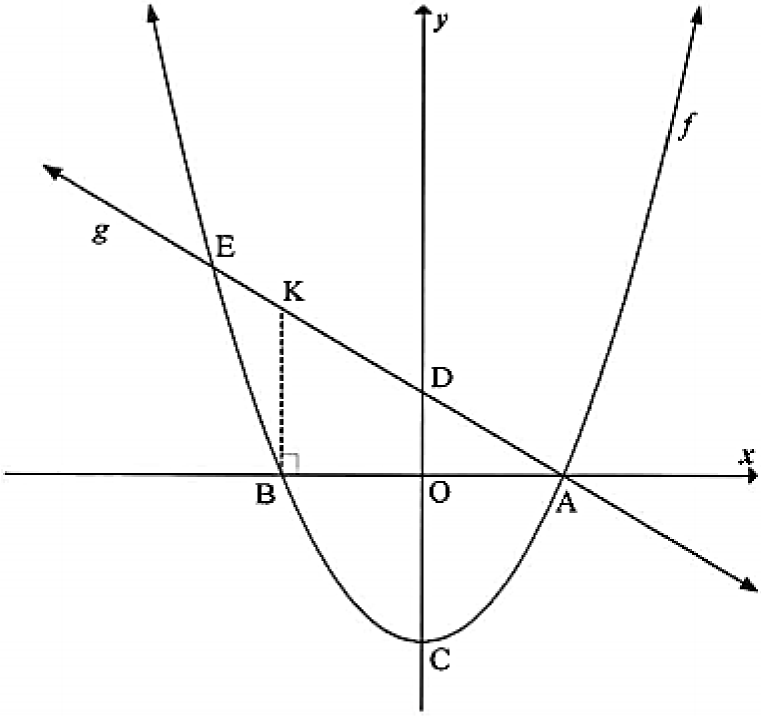


For $f(x) > 0$ or $f(x) < 0$	For $f(x) > g(x)$ or $f(x) < g(x)$	For $f(x).g(x) > 0$ or $f(x).g(x) < 0$
Focus on the $x - axis$. $f(x) > 0$ means where the graph of $f(x)$ is positive, which will be above the $x - axis$. And $f(x) < 0$ means where the graph of $f(x)$ is negative, which will be below the $x - axis$.	$f(x) > g(x)$ means where the graph of $f(x)$ is above the graph of $g(x)$. And $f(x) < g(x)$ means where the graph of $f(x)$ is below the graph of $g(x)$.	**$f(x).g(x) > 0$ means where the product of $f(x)$ and $g(x)$ is positive.** And **$f(x).g(x) < 0$ means where the product of $f(x)$ and $g(x)$ is negative.**

Example 1

QUESTION 5

The graphs of $f(x)=x^2-4$ and $g(x)=-x+2$ are sketched below. A and B are the x-intercepts of f. C and D are the y-intercepts of f and g respectively. K is a point on g such that BK $\parallel$ x-axis. f and g intersect at A and E.



5.1 Write down the coordinates of C.

5.2 Write down the coordinates of D.

5.3 Determine the length of CD.

5.4 Calculate the coordinates of B.

5.5 Determine the coordinates of E, a point of intersection of f and g.

5.6 For which values of x will:

5.6.1 $f(x) < g(x)$

5.6.2 $f(x).g(x) \geq 0$

5.7 Calculate the length of AK.

Solutions

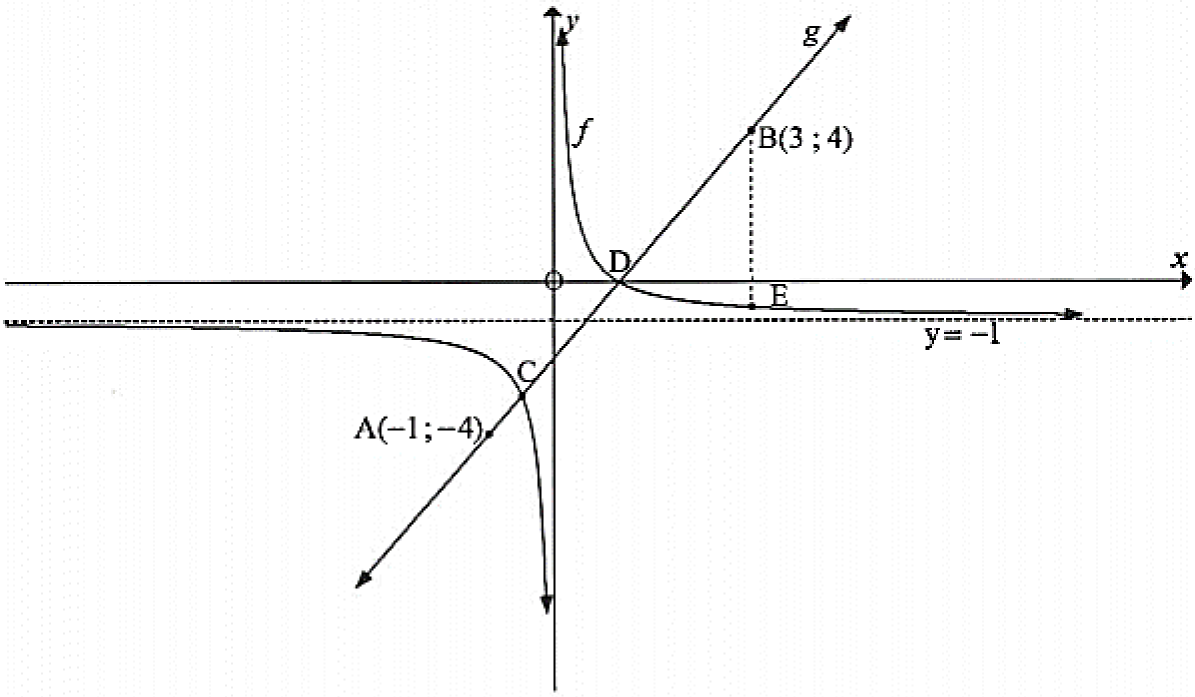
QUESTION 5	
5.1	C(0 ; –4)
5.2	D(0 ; 2)
5.3	CD = 2 – (–4) CD = 6 units/*eenhede*
5.4	$x^2 - 4 = 0$ $(x-2)(x+2) = 0$ $x = 2 \quad x = -2$ B(–2 ; 0)
5.5	$x^2 - 4 = -x + 2$ $x^2 + x - 6 = 0$ $(x-2)(x+3) = 0$ $x = 2 \quad x = -3$ E(–3 ; 5)
5.6.1	$-3 < x < 2$ **OR/*OF*** $(-3 ; 2)$
5.6.2	$(-\infty ; -2] \cup \{2\}$
5.7	K(–2 ; 4) BK = 4 units/*eenhede* AB = 4 units/*eenhede* AK $= \sqrt{4^2 + 4^2}$ (Pythagoras) = 5,66 units/*eenhede*

Example 2

QUESTION 5

The sketch below shows f and g, the graphs of $f(x)=\frac{1}{x}-1$ and $g(x)=ax+q$ respectively.

Points A(–1 ; –4) and B(3 ; 4) lie on the graph g.
The two graphs intersect at points C and D.
Line BE is drawn parallel to the y-axis, with E on f.



5.1 Show that $a=2$ and $q=-2$.

5.2 Determine the values of x for which $f(x)=g(x)$.

5.3 For what values of x is $g(x)\geq f(x)$?

5.4 Calculate the length of BE.

5.5 Write down an equation of h if $h(x)=f(x)+3$.

Solutions

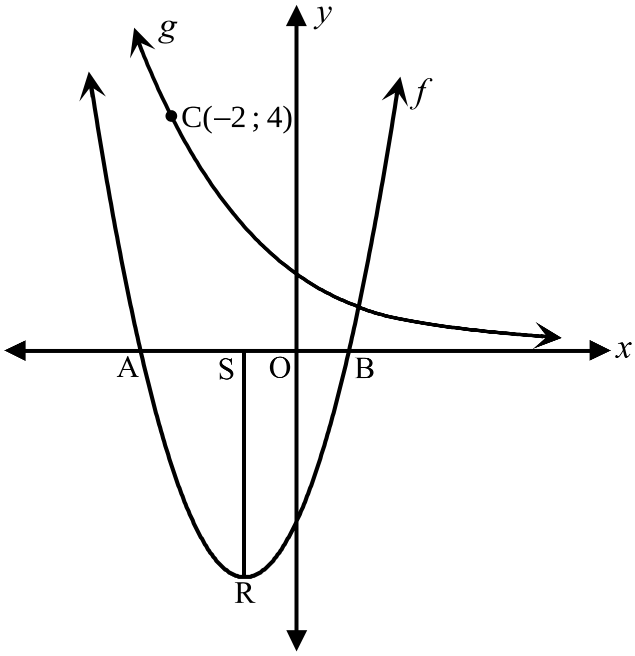
5.1	$a = \text{gradient of } g$ $= \frac{-4-4}{-1-3}$ $= 2$ $4 = 2(3) + q$ $q = -2$ $g(x) = 2x - 2$
5.2	$\frac{1}{x} - 1 = 2x - 2$ $\frac{1}{x} = 2x - 1$ $1 = 2x^2 - x$ $2x^2 - x - 1 = 0$ $(2x+1)(x-1) = 0$ $x = -\frac{1}{2}$ or $x = 1$

5.3	$-\frac{1}{2} \leq x < 0$ or/*of* $x \geq 1$ **OR/OF** $\left[-\frac{1}{2}\,;\,0\right) \cup [1\,;\,\infty)$
5.4	$f(3) = \frac{1}{3} - 1$ $= -\frac{2}{3}$ Length of BE $= 4 - f(3)$ $= 4 - \left(-\frac{2}{3}\right)$ $= 4 + \frac{2}{3}$ $= 4\frac{2}{3}$ **OR/OF** $\text{BE} = 2x - 2 - \frac{1}{x} + 1$ $= \frac{2x^2 - x - 1}{x}$ $(x = 3)\ \text{BE} = \frac{2(3)^2 - (3) - 1}{3}$ $= \frac{18 - 4}{3}$ $= 4\frac{2}{3}$
5.5	$h(x) = f(x) + 3$ $h(x) = \frac{1}{x} + 2$

Example 3

QUESTION 1

The graphs of $f(x) = 2x^2 + 4x - 6$ and $g(x) = a^x$ are represented in the sketch below. A and B are the *x*-intercepts of f and R is the turning point of f. The point $C(-2\,;4)$ is a point on the graph of *g*.



1.1 Show that $a = \frac{1}{2}$.

1.2 Determine the length of AB.

1.3 Determine the length of SR.

1.4 Write down the equation of *h*, if *h* is the reflection of f in the *y*-axis. Express your answer in the form $h(x) = a(x+p)^2 + q$.

1.5 Write down the equation of g^{-1} in the form $y = ...$

1.6 Sketch the graph of $y = g^{-1}(x)$ on a set of axes.

1.7 Determine the values of *x* for which:

1.7.1 $g^{-1}(x) \geq -2$

Solutions

1.1

$$y = a^x$$

$$\therefore 4 = a^{-2}$$

$$\therefore 4 = \frac{1}{a^2}$$

$$\therefore 4a^2 = 1$$

$$\therefore a^2 = \frac{1}{4}$$

$$\therefore a = \frac{1}{2}$$

1.2

$$0 = 2x^2 + 4x - 6$$

$$\therefore 0 = x^2 + 2x - 3$$

$$\therefore 0 = (x+3)(x-1)$$

$$\therefore x = -3 \quad \text{or} \quad x = 1$$

$$\therefore \text{AB} = 4 \text{ units}$$

1.3

$$x_{\text{R}} = -\frac{4}{2(2)} = -1$$

$$\therefore y_{\text{R}} = 2(-1)^2 + 4(-1) - 6 = -8$$

$$\therefore \text{SR} = 8 \text{ units}$$

Alternatively:

$$f'(x) = 4x + 4$$

$$\therefore 0 = 4x + 4$$

$$\therefore x = -1$$

1.4

$$h(x) = 2(x-1)^2 - 8$$

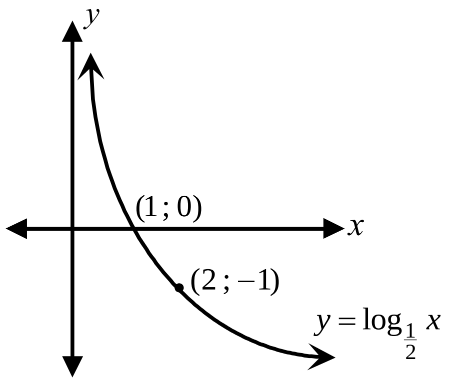
1.5

$$y = \left(\frac{1}{2}\right)^x$$

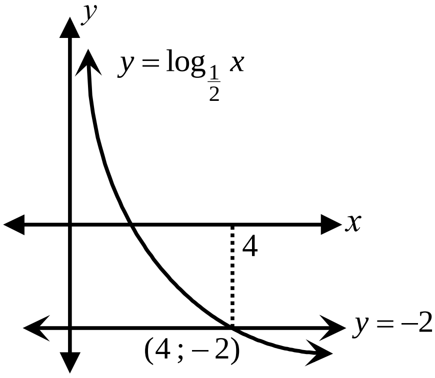
$$\therefore x = \left(\frac{1}{2}\right)^y$$

$$\therefore y = \log_{\frac{1}{2}} x$$

1.6



1.7.1



$$\log_{\frac{1}{2}} x=-2$$

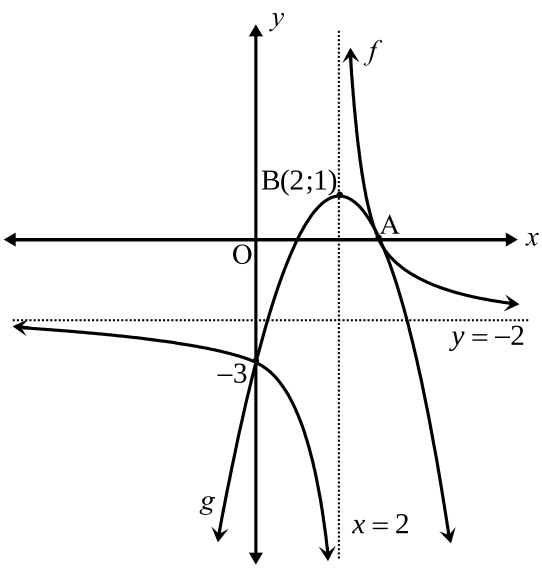
$$\therefore x=\left(\frac{1}{2}\right)^{-2}=4$$

$$\therefore g^{-1}(x)\geq -2 \text{ for all } 0<x\leq 4$$

Example 4

QUESTION 1

In the diagram below, the graph of $f(x)=\dfrac{a}{x+p}+q$ cuts the y-axis at -3 and the x-axis at A. The graph of $g(x)=m(x+n)^2+c$ intersects f at A, cuts the y-axis at -3 and has a turning point at B(2;1).



Determine:

1.1 the equation of *f*.

1.2 the equation of *g*.

1.3 the length of OA.

1.4 the values of *x* for which $f(x).g(x) \leq 0$.

Solutions

1.1

$$y = \frac{a}{x-2} - 2$$

Substitute $(0\,;-3)$:

$$-3 = \frac{a}{0-2} - 2$$

$$\therefore -1 = \frac{a}{-2}$$

$$\therefore a = 2$$

$$\therefore f(x) = \frac{2}{x-2} - 2$$

1.2

$$y = m(x-2)^2 + 1$$

Substitute $(0\,;-3)$:

$$\therefore -3 = m(0-2)^2 + 1$$

$$\therefore -4 = m(-2)^2$$

$$\therefore -4 = 4m$$

$$\therefore m = -1$$

$$\therefore g(x) = -(x-2)^2 + 1$$

1.3

$$0 = \frac{2}{x-2} - 2$$

$$\therefore 0 = 2 - 2(x-2)$$

$$\therefore 0 = 2 - 2x + 4$$

$$\therefore 2x = 6$$

$$\therefore x = 3$$

$$\therefore \text{OA} = 3 \text{ units}$$

Alternatively:

$0 = -(x-2)^2 + 1$

$\therefore (x-2)^2 = 1$

$\therefore x^2 - 4x + 4 = 1$

$\therefore x^2 - 4x + 3 = 0$

$\therefore (x-1)(x-3) = 0$

$x = 1$ or $x = 3$

$\therefore \text{OA} = 3$ units

1.4 $f(x).g(x) \leq 0$ for all

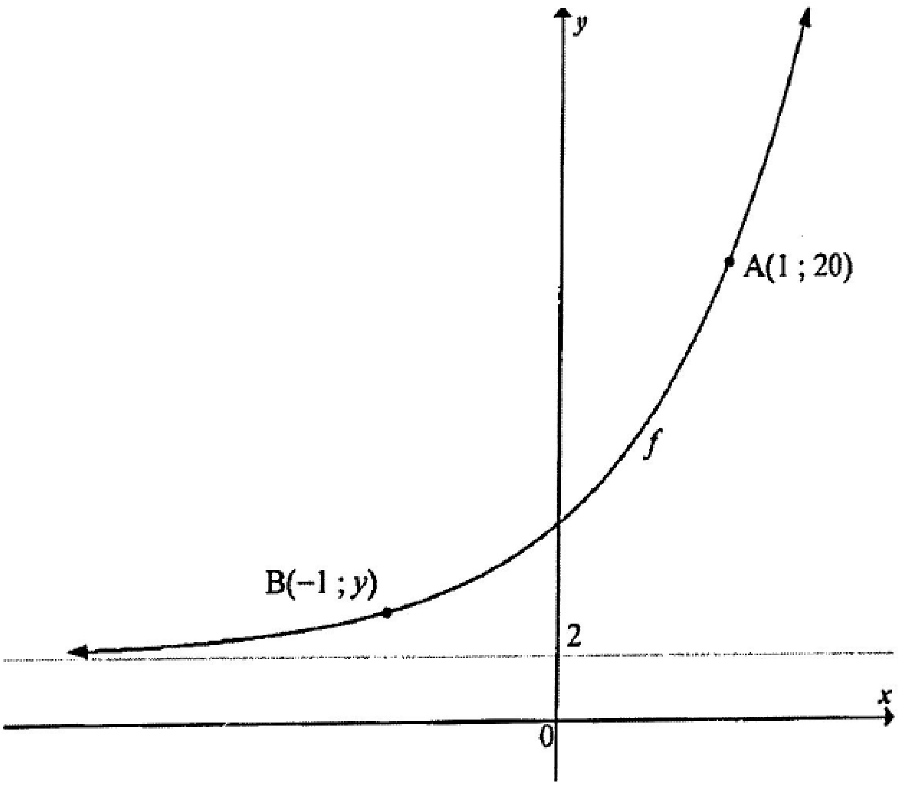
$1 \leq x < 2$ or $x = 3$

THE AVERAGE GRADIENT BETWEEN TWO POINTS

The **average gradient** of a function between any two points is defined to be the **gradient of the line** joining the two points.

Example

The sketch below is the graph of $f(x) = 2.b^{x+1} + q$.
The graph of f passes through the points A(1 ; 20) and B (−1 ; y).
The line $y = 2$ is an asymptote of f.



1. Show that the equation of f is $f(x) = 2(3)^{x+1} + 2$

2. Calculate the y-coordinate of the point B.

3. Determine the average gradient of the curve between the points A and B.

Solutions

1.
$$q = 2$$
$$f(x) = 2.b^{x+1} + 2$$
$$20 = 2.b^{1+1} + 2$$
$$18 = 2.b^2$$
$$9 = b^2$$
$$b = 3$$
$$f(x) = 2.3^{x+1} + 2$$

2.
$$y = 2.3^{-1+1} + 2$$
$$y = 2.1 + 2$$
$$y = 4$$

3.
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{20-4}{1-(-1)}$$
$$= 8$$

Activities

1. a) Determine the average gradient of the graph of $y = 5x^2 - 4$ between $x = -4$ and $x = -1$
   b) Is the function increasing or decreasing between $x = -4$ and $x = -1$?
2. Determine the average gradient of the graph of $y = 5x^2 - 4$ between:
   a) $x = 1$ and $x = 3$
   b) $x = 2$ and $x = 3$
   c) $x = 2{,}5$ and $x = 3$
   d) $x = 2{,}99$ and $x = 3$
3. a) Calculate the average gradient of the curve $f(x) = x(x + 3)$ between $x = 5$ and $x = 3$.
   b) What can you deduce about the function f between $x = 5$ and $x = 3$?

SOLUTIONS TO ACTIVITIES

Hyperbolic Functions

1.1 $y=1$ is the horizontal asymptote.

Now substitute $y=1$ into $y=-x+2$:

$\therefore 1=-x+2$

$\therefore x=1$

$\therefore$ B is the point $(1;1)$

The vertical asymptote's equation is $x=1$

Therefore, the equation of the hyperbola is

$f(x)=\dfrac{-4}{x-1}+1$

The value of p is $p=-1$.

1.2 The equation of the other axis of symmetry is:

$y=(x+p)+q$

$\therefore y=(x-1)+1$

$\therefore y=x$

$\therefore g(x)=x$

1.3 Domain of $f: x \in R, x \neq 1$

1.4

$0=\dfrac{-4}{x-1}+1$

$\therefore 0=-4+x-1$

$\therefore x=5$

A$(5;0)$

If the graph of f is shifted 5 units left, then the newly formed graph's asymptote will be $x=-4$.

Quadratic Functions

1.1.1 Intercepts with the axes:

y-intercept: Let $x=0$

$y=-(0-2)^2+1=-3$

$(0\,;-3)$

x-intercepts: Let $y=0$

$0=-(x-2)^2+1$

$\therefore (x-2)^2=1$

$\therefore x^2-4x+4=1$

$\therefore x^2-4x+3=0$

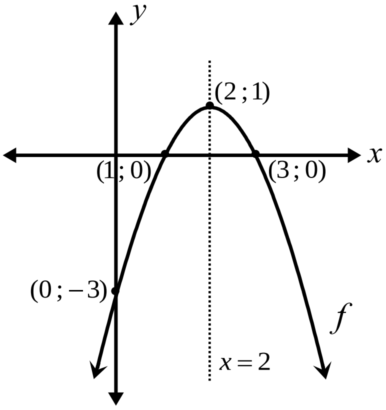
$\therefore (x-1)(x-3)=0$

$x=1$ or $x=3$

$(1\,;0)$ and $(3\,;0)$

Axis of symmetry: $x=2$

Turning point: $(2\,;1)$



1.1.2 Range: $y\in(-\infty\,;1]$

1.2.1 $f(x)=x^2-2x-3$

$x_{\text{TP}}=-\dfrac{(-2)}{2(1)}=1$

$y_{\text{TP}}=(1)^2-2(1)-3=-4$

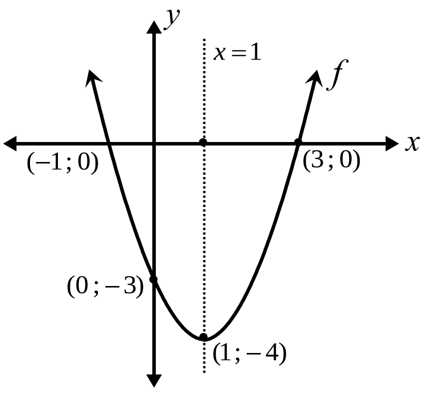
$(1;-4)$

1.2.2 $0=x^2-2x-3$

$\therefore 0=(x-3)(x+1)$

$\therefore x=3$ or $x=-1$

1.2.3



1.2.4 minimum value $=-4$.

1.2.5 $x=0$

1.2.6 $y=2(x+3)^2+1$

Exponential Functions

1.1 $y = 2^{0+1} - 1 = 1$

$(0\,;1)$

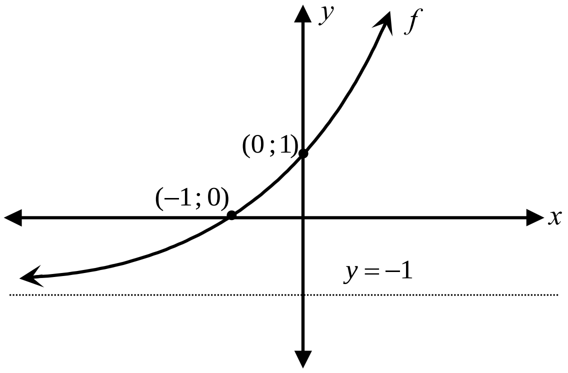
1.2 $0 = 2^{x+1} - 1$

$\therefore 1 = 2^{x+1}$

$\therefore 2^0 = 2^{x+1}$

$\therefore x = -1 \quad (-1\,;0)$

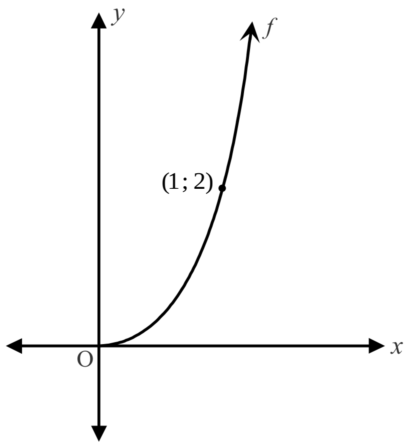
1.3



Inverse Functions

QUESTION 1

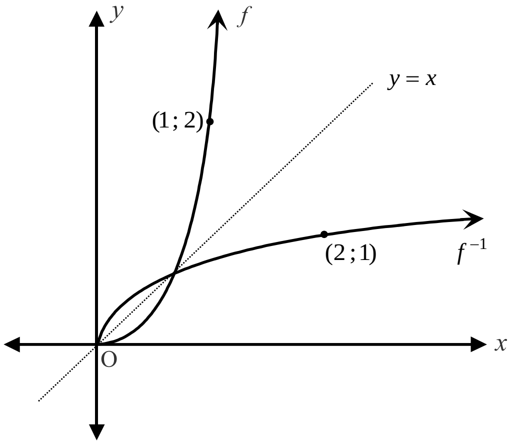
1.1



1.2

$$y = 2x^2 \quad x \geq 0 \quad f$$
$$x = 2y^2 \quad y \geq 0 \quad f^{-1}$$
$$\therefore \frac{x}{2} = y^2$$
$$\therefore y = \sqrt{\frac{x}{2}} \quad (x \geq 0)$$

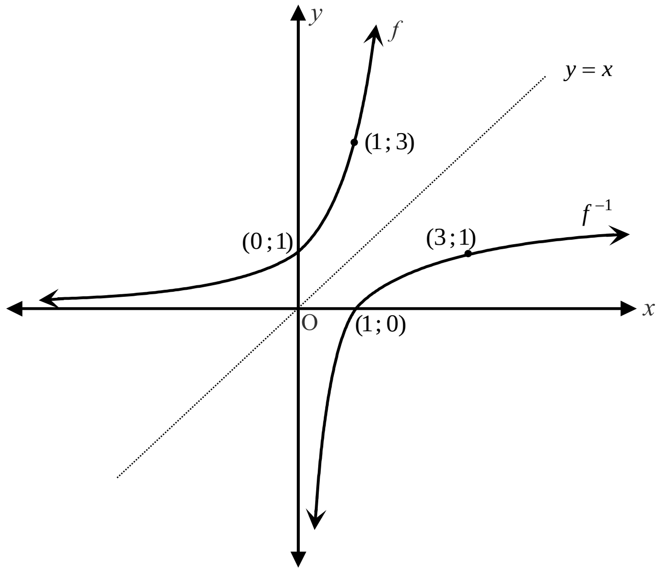
1.3



QUESTION 2

2.1 $y = \log_3 x$

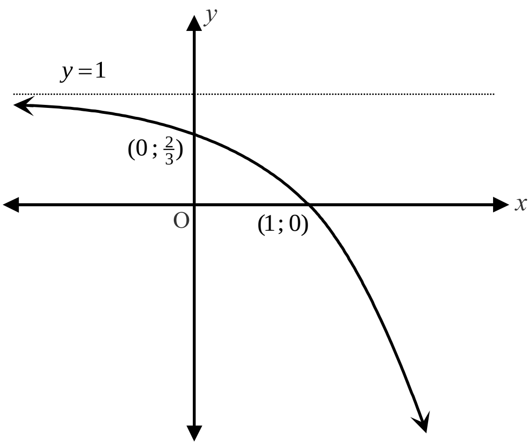
2.2



2.3.1 $f(x).f^{-1}(x) \leq 0$ for all $0 < x \leq 1$

2.3.2 $x.f^{-1}(x) > 0$ for all $x > 1$.

2.4 $y = g(x) = -3^{x-1} + 1$



Average Gradient Between Two Points

1. a) At $x = -4$

$y = 5(-4)^2 - 4 = 80 - 4 = 76$

At $x = -1$

$y = 5(-1)^2 - 4 = 5 - 4 = 1$

$\frac{y_2 - y_1}{x_2 - x_1} = \frac{76 - 1}{-4 - (-1)} = \frac{75}{-3} = -25$ (2)

b) The function is decreasing between $x = -4$ and $x = -1$ because the gradient is negative.

2. a) The points at $x = 1$ and $x = 3$ are (1; 1) and (3; 41)

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{41 - 1}{3 - 1} = \frac{40}{2} = 20$ (2)

b) The points at $x = 2$ and $x = 3$ are (2; 16) and (3; 41)

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{41 - 16}{3 - 2} = \frac{25}{1} = 25$

c) The points at $x = 2{,}5$ and $x = 3$ are (2,5; 27,25) and (3; 41)

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{41 - 27.25}{3 - 2.5} = \frac{13.75}{0.5} = 27.5$ (2)

d) The points at $x = 2{,}99$ and $x = 3$ are (2,99; 40,7) and (3; 41)

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{41 - 40.7}{3 - 2.99} = \frac{0.3}{0.01} = 30$

3. a) The points are (5; 40) and (3; 18).

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{18 - 40}{3 - 5} = \frac{-22}{-2} = 11$ (2)

b) The function is increasing between $x = 5$ and $x = 3$.

Appendix A: Examination Guidelines

ELABORATION OF CONTENT/TOPICS

The purpose of the clarification of the topics is to give guidance to the teacher in terms of depth of content necessary for examination purposes. Integration of topics is encouraged as learners should understand Mathematics as a holistic discipline. Thus questions integrating various topics can be asked.

1. Candidates must be able to use and interpret functional notation. In the teaching process learners must be able to understand how $f(x)$ has been transformed to generate $f(-x)$, $-f(x)$, $f(x+a)$ $f(x)+a$, $af(x)$ and $x=f(y)$ where $a \in R$.
2. Trigonometric functions will ONLY be examined in PAPER 2.

Appendix B: Information Sheet

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$A=P(1+ni)$ $\quad$ $A=P(1-ni)$ $\quad$ $A=P(1-i)^n$ $\quad$ $A=P(1+i)^n$

$T_n=a+(n-1)d$ $\quad$ $S_n=\frac{n}{2}\left[2a+(n-1)d\right]$

$T_n=ar^{n-1}$ $\quad$ $S_n=\frac{a\left(r^n-1\right)}{r-1}$; $r\neq 1$ $\quad$ $S_\infty=\frac{a}{1-r}$; $-1<r<1$

$F=\frac{x\left[(1+i)^n-1\right]}{i}$ $\quad$ $P=\frac{\left[1-(1+i)^{-n}\right]}{i}$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ $\quad$ $\mathrm{M}\left(\frac{x_1+x_2}{2};\frac{y_1+y_2}{2}\right)$

$y=mx+c$ $\quad$ $y-y_1=m(x-x_1)$ $\quad$ $m=\frac{y_2-y_1}{x_2-x_1}$ $\quad$ $m=\tan\theta$

$$(x-a)^2+(y-b)^2=r^2$$

In ΔABC: $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$ $\quad$ $a^2=b^2+c^2-2bc.\cos A$ $\quad$ $area\ \Delta ABC=\frac{1}{2}ab.\sin C$

$\sin(\alpha+\beta)=\sin\alpha.\cos\beta+\cos\alpha.\sin\beta$ $\quad$ $\sin(\alpha-\beta)=\sin\alpha.\cos\beta-\cos\alpha.\sin\beta$

$\cos(\alpha+\beta)=\cos\alpha.\cos\beta-\sin\alpha.\sin\beta$ $\quad$ $\cos(\alpha-\beta)=\cos\alpha.\cos\beta+\sin\alpha.\sin\beta$

$\cos 2\alpha=\begin{cases}\cos^2\alpha-\sin^2\alpha\\ 1-2\sin^2\alpha\\ 2\cos^2\alpha-1\end{cases}$ $\quad$ $\sin 2\alpha=2\sin\alpha.\cos\alpha$

$\bar{x}=\frac{\sum fx}{n}$ $\quad$ $\sigma^2=\frac{\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2}{n}$

$P(A)=\frac{n(A)}{n(S)}$ $\quad$ $P(A \text{ or } B)=P(A)+P(B)-P(A \text{ and } B)$

$\hat{y}=a+bx$ $\quad$ $b=\frac{\sum(x-\bar{x})(y-\bar{y})}{\sum(x-\bar{x})^2}$

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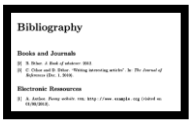
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